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Wide Morita contexts and equivalences of comodule categories

F. Castaño Iglesias [a,1], J. Gómez Torrecillas [b,*,2]

[a] Departamento de Estadística y Matemática Aplicada, Universidad de Almería, E04120 La Cañada (Almería), Spain

[b] Departamento de Algebra, Facultad de Ciencias, Universidad de Granada, E18071 Granada, Spain

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Abstract

We introduce a purely categorical notion of Morita context between abelian categories, which extends the classical notion of Morita context for module categories as well as the Takeuchi's notion of Morita context for comodule categories (Takeuchi, 1977). We prove an extension both of Morita's theorem on categories of modules and of Takeuchi's theorem on comodule categories for our general notion of Morita context. As an application of our theory, we obtain an equivalence between certain comodule categories defined by any Takeuchi's Morita context similar to that obtained by Nicholson–Watters for module categories at (Nicholson and Watters, 1988). © 1998 Elsevier Science B.V. All rights reserved.

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0. Introduction

The classical Morita's theorem on equivalence of categories of modules can be performed upon the notion of Morita context (see [3]). From this viewpoint, the theorem says that if $(R, S, {}_RV_S, {}_SW_R)$ is a Morita context between unitary rings R and S, then the functors $V \otimes_S -$ and $W \otimes_R -$ form an equivalence between the categories of unital modules R-Mod and S-Mod if and only if the trace ideals of the context are the whole rings. These conditions entail strong restrictions on the Morita context. There are some generalizations of this theorem. The common idea is to obtain an equivalence between certain categories related to R-Mod and S-Mod for a Morita context without restrictions.

* Corresponding author. E-mail: torrecil@ugr.es.

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For example, it was shown in [4] that the "hom" functors associated to the context induce an equivalence between appropriate quotient categories of R-Mod and S-Mod. A different approach was given in [6], where it was proved that suitable modifications of the functors $V\otimes_S -$ and $W\otimes_R -$ provide an equivalence between the subcategories ${}_R\mathrm{C}$ and ${}_S\mathrm{C}$ of R-Mod and S-Mod consisting of the trace-torsion-free trace-accessible modules. This theorem was the motivation of our work [2], where we proposed a purely functorial notion of Morita context between Grothendieck categories, and we proved a generalization of [6, Theorem 5] for the there-called wide Morita contexts.

On the other hand, there is a version of the Morita's theorem on equivalence for categories of comodules over coalgebras, stemming from a notion of Morita context between two coalgebras (see [9]). Of course, the fact that the Morita–Takeuchi context provides an equivalence between the whole categories of comodules via the cotensor functors forces that the context is strict in the sense of [9]. In this paper we will show, for an arbitrary Morita-Takeuchi context, the existence of an equivalence between appropriate subcategories of the categories of comodules throughout certain modifications of the cotensor product functors (see Theorem 4.2). To do this, we introduce the notion of wide left Morita context between abelian categories. This is a dualization of the notion introduced in [2] for Grothendieck categories. This last concept will be called here wide right Morita context. We have extended our notions from Grothendieck categories to abelian categories because in this framework, we can use duality ideas to give an interpretation for wide right Morita contexts of any result on wide left Morita contexts. In fact, the main theorem of Section 2 (Theorem 2.4) improve the main theorem of [2] (see Section 3 and Theorem 3.1).

1. Preliminaries

If $f:A \to B$ and $g:B \to C$ are morphisms in a category, then the composite arrow will be denoted by $g\circ f$ or, sometimes, by gf. Keep the same for the composition of functors. By the word functor we refer to a covariant functor.

For a category $\mathscr{A}$, and objects A and B of $\mathscr{A}$, the set of all morphisms from A to B will be denoted by $\mathrm{Hom}_{\mathscr{A}}(A,B)$. With this notation, the opposite or dual category of $\mathscr{A}$ is the category $\mathscr{A}^{\mathrm{opp}}$ whose objects are the objects of $\mathscr{A}$ but $\mathrm{Hom}_{\mathscr{A}^{\mathrm{opp}}}(A,B) = \mathrm{Hom}_{\mathscr{A}}(B,A)$. Of course, the composition law of $\mathscr{A}^{\mathrm{opp}}$ is the composition of $\mathscr{A}$ in the reversed order. Every functor $F:\mathscr{A} \to \mathscr{B}$ can be considered as well as a functor $F:\mathscr{A}^{\mathrm{opp}} \to \mathscr{B}^{\mathrm{opp}}$. The notion of complete, cocomplete and locally small abelian category is that of [7]. Categories of this type are the Grothendieck categories (the completeness can be deduced from the Gabriel–Popescu Theorem [7, Theorem X.4.1 and Corollary X.4.4]).

Now, we will recall briefly some facts about "torsion theory". A detailed amount can be found in [7, Ch. VI]. A *preradical* r of an abelian category $\mathscr{A}$ is a subfunctor of the identity functor $1_{\mathscr{A}}$. An object A of $\mathscr{A}$ is said to be *r-torsion free* (resp. *r-torsion*) if $r(A)=0$ (resp. $r(A)=A$). The full subcategory of $\mathscr{A}$ consisting of all the r-torsion-free (resp. r-torsion) objects of $\mathscr{A}$ will be denoted by $\mathscr{F}(r)$ (resp. $\mathscr{T}(r)$). If A is r-torsion and B is r-torsion free, then $\mathrm{Hom}_{\mathscr{A}}(A,B)=0$. A preradical r is said to be *idempotent* if $r(r(A))=r(A)$ for every object A of $\mathscr{A}$. A preradical is a *radical* if $r(A/r(A))=0$ for every object A of $\mathscr{A}$. Assume that $\mathscr{A}$ is complete, cocomplete and locally small. A class C of objects of $\mathscr{A}$ is said to be a *pretorsion class* (resp. *pretorsion-free class*) if C is closed under quotient objects and coproducts (resp. subobjects and products). If, in addition, C is closed under extensions, we speak of torsion (resp. torsion-free) classes. The assignment $r \mapsto \mathscr{T}(r)$ provides a bijective correspondence between the pretorsion classes of $\mathscr{A}$ and the idempotent preradicals of $\mathscr{A}$. Dually, the assignment $r \mapsto \mathscr{F}(r)$ is a bijective correspondence between the pretorsion-free classes of $\mathscr{A}$ and the radicals of $\mathscr{A}$. These correspondences specialize to bijective correspondences between idempotent radicals and torsion (resp. torsion-free) classes. In particular, there is a bijective correspondence between torsion and torsion-free classes.

Let C be a full subcategory of an abelian category $\mathscr{A}$. An injective object E of $\mathscr{A}$ *cogenerates* C if $\mathrm{Hom}_{\mathscr{A}}(C,E) \neq 0$ for every nonzero object C of C. Analogously, a projective object P *generates* C if $\mathrm{Hom}_{\mathscr{A}}(P,C) \neq 0$ for every object C of C.

2. Equivalences of categories

Consider $\mathscr{A}$ and $\mathscr{B}$ abelian categories. A *wide left Morita context* for $\mathscr{A}$ and $\mathscr{B}$ is a pair of left exact functors

$$G : \mathscr{A} \rightleftarrows \mathscr{B} : F$$

together with natural transformations $\eta : 1_{\mathscr{A}} \to FG$ and $\rho : 1_{\mathscr{B}} \to GF$ that satisfy the following *compatibility conditions*:

- $G(\eta_A) = \rho_{GA}$ for every $A \in \mathscr{A}$.
- $F(\rho_B) = \eta_{FB}$ for every $B \in \mathscr{B}$.

The wide left Morita context will be denoted by $(\mathscr{A},\mathscr{B},G,F,\eta,\rho)$. Now we will consider idempotent radicals $r : \mathscr{A} \to \mathscr{A}$ and $s : \mathscr{B} \to \mathscr{B}$, such that $G(\mathscr{F}(r)) \subseteq \mathscr{F}(s)$ and $F(\mathscr{F}(s)) \subseteq \mathscr{F}(r)$. In this case we will say that

$$\mathbf{S} = (\mathscr{A},\mathscr{B},G,F,\eta,\rho,r,s)$$

is a *left preequivalence situation.*

Define $G^* = sG$ and $F^* = rF$, and consider, for $A \in \mathscr{A}$ and $B \in \mathscr{B}$, the canonical injections $\alpha_A : G^*A \to GA$ and $\beta_B : F^*B \to FB$. In this way we have natural transformations $\alpha : G^* \to G$ and $\beta : F^* \to F$. Observe that G^* and F^* preserve monomorphisms.

Proposition 2.1. *Let $inc_{\mathscr{T}(r)} : \mathscr{T}(r) \to \mathscr{A}$ and $inc_{\mathscr{T}(s)} : \mathscr{T}(s) \to \mathscr{B}$ denote the inclusion functors. There are natural transformations*

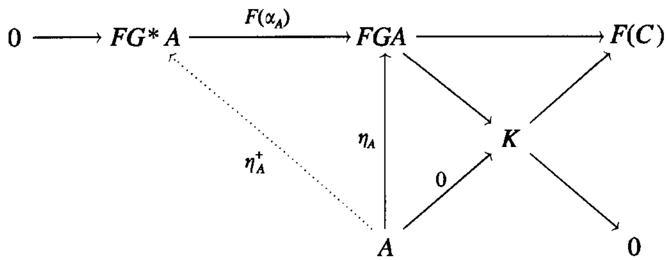
$$\eta^+ : inc_{\mathscr{T}(r)} \to FG^*$$

and

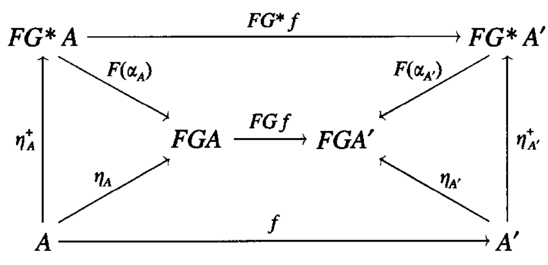
$$\rho^+ : inc_{\mathcal{T}(s)} \to GF^*$$

such that $\eta_A = F(\alpha_A) \circ \eta_A^+$ *for every r-torsion object A of* $\mathcal{A}$ *and* $\rho_B = G(\beta_B) \circ \rho_B^+$ *for every s-torsion object B of* $\mathcal{B}$.

Proof. We need only to prove the existence of η^+. Given a r-torsion object A of $\mathcal{A}$, consider $GA \to C$ the cokernel of the monomorphism $\alpha_A : G^*A \to GA$. Since F is a left exact functor, we have the following commutative diagram with exact upper row



where $FGA \to K$ is the cokernel of $F(\alpha_A)$ and, so, $F(\alpha_A)$ is its kernel. Moreover, the exactness of the upper row guarantees the existence of the embedding $K \to F(C)$. Since C is s-torsion-free, we have that $F(C)$ is r-torsion-free, whence K is r-torsion-free. This entails that any homomorphism from A to K has to be the zero map. Therefore, η_A^+ is uniquely determined by the condition $F(\alpha_A) \circ \eta_A^+ = \eta_A$. Now we will show that the foregoing construction gives a natural transformation $\eta^+ : inc_{\mathcal{T}(r)} \to FG^*$. To check the naturality, take any $\mathcal{A}$-morphism $f : A \to A'$ and consider the diagram



with commutative triangles and trapezia. The fact that $F(\alpha_{A'})$ is a monomorphism allows to argue on this diagram to obtain that the rectangle commutes, that is, $\eta_{A'}^+ \circ f = FG^*f \circ \eta_A^+$. □

Proposition 2.2. *There are natural transformations*

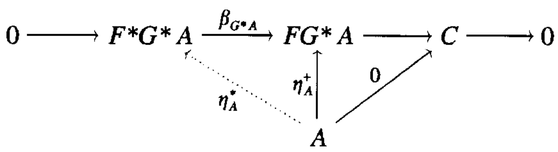
$$\eta^* : 1_{\mathcal{T}(r)} \to F^*G^*$$

and

$$\rho^* : 1_{\mathcal{T}(s)} \to G^*F^*$$

such that $\eta_A^+ = \beta_{G^*A} \circ \eta_A^*$ *for every r-torsion object A of* $\mathscr{A}$ *and* $\rho_B^+ = \alpha_{F^*B} \circ \rho_B^*$ *for every s-torsion object B of* $\mathscr{B}$.

Proof. Given an r-torsion object A of $\mathscr{A}$, let $FG^*A \to C$ be the cokernel of the monomorphism $\beta_{G^*A} : F^*G^*A \to FG^*A$. Since C is s-torsionfree, the only morphism from A to C is the zero morphism. This implies that there exists a unique morphism $\eta_A^* : A \to F^*G^*A$ making the following diagram commutative:



This construction defines a natural transformation $\eta^* : 1_{\mathscr{T}(r)} \to F^*G^*$. The proof is similar to that of Proposition 2.1. □

Remark 2.3. It is not difficult to show that the functors F^* and G^* and the natural transformations η^* and ρ^* satisfy the following relations:
- $G^*(\eta_A^*) = \rho_{G^*A}^*$ for every object A in $\mathscr{T}(r)$.
- $F^*(\rho_B^*) = \eta_{F^*B}^*$ for every object B in $\mathscr{T}(s)$.

Our aim is to give a sufficient condition on the preequivalence data to obtain that G^* and F^* form an equivalence between certain full subcategories of $\mathscr{A}$ and $\mathscr{B}$. We will say that an object A of $\mathscr{A}$ is *η-coaccessible* if η_A is a monomorphism. Analogously, we can define *ρ-coaccessible* objects $\mathscr{B}$. For a preequivalence situation **S** we will denote by $\mathscr{A}(\mathbf{S})$ the full subcategory of $\mathscr{A}$ whose objects are the η-coaccesible and r-torsion ones. Analogously, $\mathscr{B}(\mathbf{S})$ will be the full subcategory of $\mathscr{B}$ consisting of the ρ-coaccessible objects and s-torsion objects.

Theorem 2.4. *Assume that there are injective objects E in* $\mathscr{A}$ *and W in* $\mathscr{B}$ *such that E cogenerates every r-torsion object of* $\mathscr{A}$ *and W cogenerates every s-torsion object of* $\mathscr{B}$. *If* $\operatorname{Ker}(\eta_E)$ *is r-torsionfree and* $\operatorname{Ker}(\rho_W)$ *is s-torsion free, then the functors*

$$G^* : \mathscr{A}(\mathbf{S}) \rightleftarrows \mathscr{B}(\mathbf{S}) : F^*$$

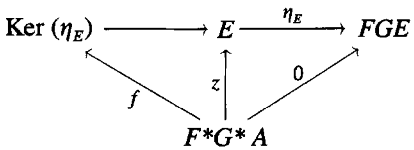
define an equivalence of categories via the natural transformations η^* *and* ρ^*.

Proof. Let A be an object of $\mathscr{A}(\mathbf{S})$. We have to prove first that G^*A belongs to $\mathscr{B}(\mathbf{S})$. Since G^*A is s-torsion, it suffices to show that it is ρ-coaccessible. But this is a consequence of the equality $GF(\alpha_A) \circ \rho_{G^*A} = \rho_{GA} \circ \alpha_A$. Now, we will prove that η_A^* is an isomorphism. From Propositions 2.1 and 2.2 it can be easily deduced that η_A^* is a monomorphism. Thus, we have only to show that $\operatorname{Coker}(\eta_A^*)=0$. Every homomorphism from $\operatorname{Coker}(\eta_A^*)$ to E is determined by $z : F^*G^*A \to E$ such that $z \circ \eta_A^* = 0$. We will prove that $z = 0$.

First, we claim that it suffices to prove that

$$\eta_E \circ z = 0. \tag{1}$$

In fact, if $\eta_E \circ z = 0$ then there is a morphism f such that the following diagram commutes:

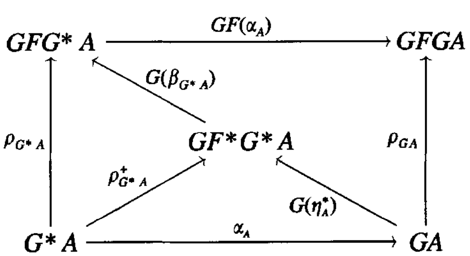


But we assume that Ker (η_E) is r-torsion free, and F^*G^*A is r-torsion, whence $f = 0$ and, so, $z = 0$.

To prove that $\eta_E \circ z = 0$ we will show first

$$G(\eta_A^*) \circ \alpha_A = \rho^+_{G^*A}. \tag{2}$$

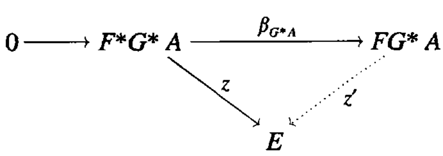
Consider the following diagram:



We have to show that the bottom triangle commutes. By Proposition 2.1, the left triangle commutes. Moreover, the square commutes by the naturality of ρ and the right triangle commutes by Propositions 2.1 and 2.2. Since $GF(\alpha_A) \circ G(\beta_{G^*A})$ is a monomorphism, we can deduce from these facts that the bottom triangle commutes. Thus, (2) holds. This gives

$$G(z) \circ \rho^+_{G^*A} = G(z) \circ G(\eta_A^*) \circ \alpha_A = G(z \circ \eta_A^*) \circ \alpha_A = 0. \tag{3}$$

Finally, let us prove (1). Since E is injective and β_{G^*A} is a monomorphism, there is $z' : FG^*A \to Z$ making the following diagram commutative:



By Proposition 2.1 and (3).

$$G(z') \circ \rho_{G^*A} = G(z') \circ G(\beta_{G^*A}) \circ \rho^+_{G^*A} = G(z' \circ \beta_{G^*A}) \circ \rho^+_{G^*A} = G(z) \circ \rho^+_{G^*A} = 0.$$

Thus, have the commutative diagram

$$\begin{array}{ccccc} & & FG^*A & \xrightarrow{F(\rho_{G^*A})=\eta_{FG^*A}} & FGFG^*A \\ & \nearrow^{\beta_{G^*A}} & \downarrow{\scriptstyle z'} & \searrow^{0} & \downarrow{\scriptstyle FG(z')} \\ F^*G^*A & \xrightarrow{z} & E & \xrightarrow{\eta_E} & FGE \end{array}$$

and we obtain that $z=0$.

We have proved that every homomorphism from $\operatorname{Coker}(\eta_A^*)$ to E is zero. Since E is a cogenerator, $\operatorname{Coker}(\eta_A^*)$ has to be zero. □

Corollary 2.5. *Let $(\mathscr{A},\mathscr{B},G,F,\eta,\rho)$ be a wide left Morita context. Assume that there are injective cogenerators E of $\mathscr{A}$ and W of $\mathscr{B}$. Then the functors*

$$G:\mathscr{A}\rightleftarrows\mathscr{B}:F$$

form an equivalence of categories via the natural transformations η and ρ if and only if η_E and ρ_W are monomorphisms.

3. Duality

Consider $\mathscr{A}$ and $\mathscr{B}$ abelian categories. A *wide right Morita context* for $\mathscr{A}$ and $\mathscr{B}$ is a pair of right exact functors

$$G:\mathscr{A}\rightleftarrows\mathscr{B}:F$$

together with natural transformations $\eta:FG\to 1_{\mathscr{A}}$ and $\rho:GF\to 1_{\mathscr{B}}$ that satisfy the following *compatibility conditions*:
- $G(\eta_A)=\rho_{GA}$ for every $A\in\mathscr{A}$,
- $F(\rho_B)=\eta_{FB}$ for every $B\in\mathscr{B}$.

This notion was introduced in [2] for $\mathscr{A}$ and $\mathscr{B}$ Grothendieck categories with the name of *wide Morita context*. We will here obtain an improved version of [2, Theorem 2.6] from Theorem 2.4. To do this, let us define a *right preequivalence situation*

$$\mathbf{S}=(\mathscr{A},\mathscr{B},G,F,\eta,\rho,r,s)$$

for the categories $\mathscr{A}$ and $\mathscr{B}$ to be a wide right Morita context

$$(\mathscr{A},\mathscr{B},G,F,\eta,\rho)$$

together with idempotent radicals $r:\mathscr{A}\to\mathscr{A}$ and $s:\mathscr{B}\to\mathscr{B}$ such that $G(\mathscr{T}(r))\subseteq\mathscr{T}(s)$ and $F(\mathscr{T}(s))\subseteq\mathscr{T}(r)$.

If $\mathscr{A}$ is a category, then its opposite category $\mathscr{A}^{\mathrm{opp}}$ is also an abelian category. Any wide *right* Morita context

$$(\mathscr{A},\mathscr{B},G,F,\eta,\rho)$$

gives a wide *left* Morita context

$$(\mathscr{A}^{\mathrm{opp}}, \mathscr{B}^{\mathrm{opp}}, G, F, \eta, \rho).$$

On the other hand, given an idempotent radical $r : \mathscr{A} \to \mathscr{A}$, we can define an idempotent radical $r^{\mathrm{opp}} : \mathscr{A}^{\mathrm{opp}} \to \mathscr{A}^{\mathrm{opp}}$ by putting $r^{\mathrm{opp}}(A) = A/r(A)$ for every object A in $\mathscr{A}^{\mathrm{opp}}$. It is easy to see that the r^{opp}-torsion objects of $\mathscr{A}^{\mathrm{opp}}$ are precisely the r-torsion-free objects of $\mathscr{A}$, and the r^{opp}-torsion-free objects of $\mathscr{A}^{\mathrm{opp}}$ are the r-torsion objects of $\mathscr{A}$. Therefore, given a right preequivalence situation

$$\mathbf{S} = (\mathscr{A}, \mathscr{B}, G, F, \eta, \rho, r, s)$$

we have got a left preequivalence situation

$$\mathbf{S}^{\mathrm{opp}} = (\mathscr{A}^{\mathrm{opp}}, \mathscr{B}^{\mathrm{opp}}, G, F, \eta, \rho, r^{\mathrm{opp}}, s^{\mathrm{opp}}).$$

Fix the right preequivalence situation $\mathbf{S}$ and let us say that an object A of $\mathscr{A}$ is η-accessible whenever η_A is an epimorphism. Consider $\mathscr{A}(\mathbf{S})$ the full subcategory of $\mathscr{A}$ whose objects are the r-torsion-free η-accessible ones. Analogously, we define the full subcategory $\mathscr{B}(\mathbf{S})$ of $\mathscr{B}$ consisting of the s-torsion-free ρ-accessible objects of $\mathscr{B}$.

From the functors G and F we can define $G^* : \mathscr{A} \to \mathscr{B}$ and $F^* : \mathscr{B} \to \mathscr{A}$ by putting $G^*A = GA/s(GA)$ for every object A in $\mathscr{A}$ and $F^*B = FB/r(FB)$ for every object B in $\mathscr{B}$. In fact, the definitions can be done with respect the opposite wide left Morita context $\mathbf{S}^{\mathrm{opp}}$. With this formalism, we can apply the theory developed in Section 2 to obtain natural transformations $\eta^* : F^*G^* \to 1_{\mathscr{A}(\mathbf{S})}$ and $\rho^* : G^*F^* \to 1_{\mathscr{B}(\mathbf{S})}$, which allows to rephrase Theorem 2.4 as follows.

Theorem 3.1. *Assume that $\mathscr{A}$ and $\mathscr{B}$ have projective objects P and Q, respectively, such that P generates every r-torsion-free object of $\mathscr{A}$ and Q generates every s-torsion-free object of $\mathscr{B}$. If* $\mathrm{Coker}\,(\eta_P)$ *is r-torsion and* $\mathrm{Coker}\,(\rho_Q)$ *is s-torsion, then the functors*

$$G^* : \mathscr{A}(\mathbf{S}) \rightleftarrows \mathscr{B}(\mathbf{S}) : F^*$$

define an equivalence of categories via the natural transformations η^* and ρ^*.

Of course, we can deduce the following corollary.

Corollary 3.2. *Let $(\mathscr{A}, \mathscr{B}, G, F, \eta, \rho)$ be a wide right Morita context. Assume that there are projective generators P of $\mathscr{A}$ and Q of $\mathscr{B}$. Then the functors*

$$G : \mathscr{A} \rightleftarrows \mathscr{B} : F$$

form an equivalence of categories via the natural transformations η and ρ if and only if η_P and ρ_Q are epimorphisms.

4. Morita–Takeuchi contexts

Fix a commutative field k. Let C be a coalgebra over k with comultiplication $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to k$. We refer to [8] for the details. The dual space $C^* = \mathrm{Hom}_k(C,k)$ can be canonically endowed with structure of k-algebra. The structure map for a right C-comodule M will be denoted by $\lambda_M : M \to M \otimes C$. The coalgebra C can be considered as a right C-comodule with structure map $\lambda_C = \Delta$. The right C-comodules together with the C-colinear maps between them form a Grothendieck category $\mathcal{M}^C$. In fact, $\mathcal{M}^C$ is isomorphic to a closed subcategory of the category C^*-Mod of all left modules over C^*. In particular, the C-colinear maps between C-comodules are precisely the C^*-linear maps between them.

The right C-subcomodules of C are called right coideals of C. Left coideals are defined similarly. It is not hard to see that if W is a k-vector space and X is a right C-comodule, then $W \otimes X$ is a right C-comodule with structure map $I_W \otimes \lambda_X : W \otimes X \to W \otimes X \otimes C$. Moreover, if W is a right C-comodule, then the structure map $\lambda_W : W \to W \otimes C$ becomes C-colinear.

Consider coalgebras C and D. Following [9], a C-D-bicomodule M is a left C-comodule and right D-comodule such that the C-comodule structure map $\mu_M : M \to C \otimes M$ is D-colinear or, equivalently, that the D-comodule structure map $\lambda_M : M \to M \otimes D$ is C-colinear.

Assume that X and Y are, respectively, right and left comodules over a coalgebra C. The cotensor product $X \square_C Y$ is defined as the kernel of the morphism $\lambda_X \otimes I_Y - I_X \otimes \lambda_Y = : X \otimes Y \to X \otimes C \otimes Y$. If ${}_C M_D$ is a C-D-bicomodule and ${}_D N_E$ is a D-E-bicomodule, then the comodule structure maps $\lambda_M : M \to C \otimes M$ and $\lambda_N : N \to N \otimes E$ induce the structure maps $\lambda_M \square_D I_N : M \square_D N \to (C \otimes M) \square_D N = C \otimes (M \square_D N)$ and $I_M \square_D \lambda_N : M \square_D N \to M \square_D (N \otimes E) = (M \square_D N) \otimes E$ with which $M \square_D N$ is a bicomodule. Moreover, $M \square_D N$ is a left C- and right E-subcomodule of $M \otimes N$. Consider the structure of right C-comodule on $X \otimes C$ defined by $1 \otimes \Delta$, then $X \square_C C$ is a C-subcomodule of $X \otimes C$. Moreover there is a comodule isomorphism $X \square_C C \cong X$ given by $x \otimes c \to x\varepsilon(c)$. Let δ_X be the inverse isomorphism and let us denote by $i : X \square_C C \to X \otimes C$ the inclusion map. The following diagram is commutative:

$$\begin{array}{ccc} X \square_C C & \xrightarrow{\;i\;} & X \otimes C \\ & {}^{\delta_X}\nwarrow & \uparrow {\scriptstyle \lambda_X} \\ & & X \end{array} \qquad (4)$$

The following is the notion of Morita context between coalgebras introduced in [9].

Definition 4.1. A *Morita–Takeuchi* context $(C, D, {}_C M_D, {}_D N_C, f, g)$ consists of coalgebras C and D, bicomodules ${}_C M_D$ and ${}_D N_C$, and bicolinear maps $f : C \to M \square_D N$ and

$g:D \to N\Box_C M$ making the following diagrams commute:

$$\begin{array}{ccc} M & \longrightarrow & M\,\Box_D D \\ \downarrow & & \downarrow{\scriptstyle I\,\Box\, g} \\ C\,\Box_C M & \xrightarrow{f\,\Box\, I} & M\,\Box_D N\,\Box_C M \end{array} \qquad \begin{array}{ccc} N & \longrightarrow & N\,\Box_C C \\ \downarrow & & \downarrow{\scriptstyle I\,\Box\, f} \\ D\,\Box_D N & \xrightarrow{g\,\Box\, I} & N\,\Box_C M\,\Box_D N \end{array}$$

Let us fix a Morita–Takeuchi context $(C, D, {}_CM_D, {}_DN_C, f, g)$. From [9] we know that this context defines two left exact functors

$$-\Box_C M : \mathcal{M}^C \rightleftarrows \mathcal{M}^D : -\Box_D N$$

which will be called *coinduction functors*. There are also natural transformations $\eta : I_{\mathcal{M}^C} \to -\Box_C M \Box_D N$ and $\rho : I_{\mathcal{M}^D} \to -\Box_D N \Box_C M$ defined for $X \in \mathcal{M}^C$ and $Y \in \mathcal{M}^D$ in the following way. We know the existence of natural colinear isomorphisms $\delta_X : X \to X \Box_C C$ and $\gamma_Y : Y \to Y \Box_D D$. Define

$$\eta_X = (I_X \Box_C f) \circ \delta_X$$

and

$$\rho_Y = (I_Y \Box_D g) \circ \gamma_Y.$$

The foregoing facts, together with [9, Remark 2.4], prove that

$$(\mathcal{M}^C, \mathcal{M}^D, -\Box_C M, -\Box_C N, \eta, \rho)$$

is a wide left Morita context.

Our next proposal is to prove a version for comodules of [6, Theorem 5]. Recall that a k-subspace A of a coalgebra C is said to be a *subcoalgebra* of C whenever $\Delta(A) \subseteq A \otimes A$. It is well known that a left and right coideal of C is a subcoalgebra of C. Therefore, $\operatorname{Ker} f$ and $\operatorname{Ker} g$ are subcoalgebras of C and D, respectively.

If A is a subcoalgebra of C then $A^{\perp} = \{f \in C^* : f(a) = 0\, \forall a \in A\}$ is an ideal of C^*. It is easy to show that

$$\mathcal{T}_A = \{X \in \mathcal{M}^C : A^{\perp}X = X\} \tag{5}$$

is a torsion class in $\mathcal{M}^C$. Therefore, there is an idempotent radical $r_f : \mathcal{M}^C \to \mathcal{M}^C$ such that its associated torsion class is $\mathcal{T}_{\operatorname{Ker} f}$. Analogously, we consider the idempotent radical $r_g : \mathcal{M}^D \to \mathcal{M}^D$. The cotensor functors can be modified as follows. For $X \in \mathcal{M}^C$, define $\overline{X \Box_C M} = r_g(X \Box_C M)$ and, for $Y \in \mathcal{M}^D$, define $\overline{Y \Box_D N} = r_f(Y \Box_D N)$. We will prove the following theorem.

Theorem 4.2. *Let*

$$(C, D, {}_CM_D, {}_DN_C, f, g)$$

be a Morita–Takeuchi context. Consider the full subcategories

$$\mathrm{C}^C=\{X\in\mathscr{M}^C:(\mathrm{Ann}_X((\mathrm{Ker}\ f)^\perp)=0 \textit{ and } (\mathrm{Ker}\ f)^\perp X=X\}$$

and

$$\mathrm{C}^D=\{Y\in\mathscr{M}^D:(\mathrm{Ann}_Y((\mathrm{Ker}\ g)^\perp)=0 \textit{ and } (\mathrm{Ker}\ g)^\perp Y=Y\}.$$

Then the functors

$$\overline{-\Box_C M}:\mathrm{C}^C \rightleftarrows \mathrm{C}^D:\overline{-\Box_D N}$$

form an equivalence of categories.

Write $X\wedge Y=\Delta^{-1}(X\otimes C+C\otimes Y)$, where X, Y are subspaces of C. By [8, Proposition 9.0.0.(c)], the "wedge" $\wedge$ is associative. Define recursively $\bigwedge^n X=X\wedge(\bigwedge^{n-1}X)$, for $n>0$, and $\bigwedge^0 X=0$. Put $\bigwedge^\infty X=\bigcup_{n>0}\bigwedge^n X$. If A is a subcoalgebra of C, then $\bigwedge^\omega A$ is a subcoalgebra of C for $\omega=0,1,\dots,\infty$.

For every C-comodule X, there is an associated subcoalgebra $C(X)$ of C [1, p. 129]. Now, for any subcoalgebra A of C, define

$$\mathrm{C}_A=\{X\in\mathscr{M}^C:C(X)\subseteq A\}. \tag{6}$$

It is immediate that $X\in\mathrm{C}_A$ if and only if $\lambda_X(X)\subseteq X\otimes A$. Therefore, by [5, Theorem 4.2], C_A is a closed subcategory of $\mathscr{M}^C$ and

$$\mathrm{C}_A=\{X\in\mathscr{M}^C:A^\perp X=0\}. \tag{7}$$

Lemma 4.3. *Let A be a subcoalgebra of C and let $\mathscr{F}_A$ denote the torsion-free class associated to the torsion class $\mathscr{T}_A$. Then, for any natural number n,*

$$\mathrm{C}_{\wedge^n A}\subseteq\mathscr{F}_A\subseteq\mathrm{C}_{\wedge^\infty A}.$$

Proof. Let $Y\in\mathrm{C}_{\wedge^n A}$, $X\in\mathscr{T}_A$ and $\phi\in\mathrm{Hom}_{\mathscr{M}^C}(X,Y)$. From [8, Proposition 9.0.0.(a)] we can easily deduce that

$$(A^\perp)^n\subseteq((A^\perp)^n)^{\perp\perp}=(\bigwedge^n A)^\perp.$$

Therefore,

$$X=(A^\perp)^n X=(\bigwedge^n A)^\perp X$$

and, thus, $\phi(X)\subseteq(\bigwedge^n A)^\perp Y=0$. This proves that $\phi=0$ and, therefore, $\mathrm{C}_{\wedge^n A}\subseteq\mathscr{F}_A$.

By [5, Theorem 4.2, Proposition 4.8] $\mathrm{C}_{\wedge^\infty A}$ is a hereditary torsion class closed under direct products. If we consider $\mathrm{C}_{\wedge^\infty A}$ as torsion-free class, then its associated torsion class is $\mathscr{T}_{\wedge^\infty A}$. Since torsion and torsion-free classes determine each one the other, it follows that $\mathscr{F}_{\wedge^\infty A}=\mathrm{C}_{\wedge^\infty A}$. Finally, since $\mathscr{T}_{\wedge^\infty A}\subseteq\mathscr{T}_A$, it follows that $\mathscr{F}_A\subseteq\mathscr{F}_{\wedge^\infty A}=\mathrm{C}_{\wedge^\infty A}$. □

Proof of Theorem 4.2. We will prove that $(\mathcal{M}^C, \mathcal{M}^D, -\square_C M, -\square_D N, \eta, \rho, r_f, r_g)$ is a preequivalence situation and that $\operatorname{Ker}\eta_X$ (resp. $\operatorname{Ker}\rho_Y$) is r_f-torsion free (resp. r_g-torsion free) for every $X \in \mathcal{M}^C$ (resp. $Y \in \mathcal{M}^D$). Therefore, Theorem 4.2 will be a consequence of Theorem 2.4.

First, let us prove that

$$\operatorname{Ker}\,\eta_X \in \mathrm{C}_{\operatorname{Ker}\,f} \quad \text{for every } X \in \mathcal{M}^C. \tag{8}$$

A routine computation shows that $\lambda_X(\operatorname{Ker}\,\eta_X) \subseteq X \otimes \operatorname{Ker}\,f$. This implies that $(\operatorname{Ker}\,f)^{\perp}$ $\operatorname{Ker}\,\eta_X = 0$ and, thus, (8) holds. Now, let A be any subcoalgebra of C. An easy consequence of diagram (1) is that for every $X \in \mathrm{C}_A$

$$X \square_C C = X \square_C A. \tag{9}$$

Since $(\mathcal{M}^C, \mathcal{M}^D, -\square_C M, \square_D N, \eta, \rho)$ is a wide left Morita context we have $\rho_{X\square_C M} = \eta_X \square_C I_M$ and, therefore,

$$\rho_{X\square_C M} = (I_X \square_C f \square_C I_M) \circ (\delta_X \square_C I_M).$$

By (9), $X \square_C C \square_C M = X \square_C A \square_C M$ and, thus,

$$\operatorname{Im}(\rho_{X\square_C M}) \subseteq X \square_C f(A) \square_C M. \tag{10}$$

If X, Y, Z are vector subspaces of C, then $X \subseteq Z \wedge Y$ if and only if $Y^{\perp} X \subseteq Z$. This allows to prove by induction on n that the bicomodule $f(\bigwedge^n \operatorname{Ker} f)$ belongs to $\mathrm{C}_{\wedge^{n-1}\operatorname{Ker}\,f}$ for every positive integer n. Therefore, for every right C-comodule X,

$$X \square_C f(\bigwedge^n \operatorname{Ker}\,f) \in C_{\wedge^{n-1}\operatorname{Ker}\,f}. \tag{11}$$

Now we are ready to prove that

$$\text{If } X \in \mathcal{F}_{\operatorname{Ker} f} \text{ then } X \square_C M \in \mathcal{F}_{\operatorname{Ker} g}. \tag{12}$$

Assume that X is a finite-dimensional comodule. By Lemma 4.3, $X \in \mathrm{C}_{\wedge^{\infty}\operatorname{Ker}\,f}$. By (6) $C(X) \subseteq \bigwedge^{\infty} \operatorname{Ker}\,f$. Since $C(X)$ is finite-dimensional, it follows that $C(X) \subseteq \bigwedge^n \operatorname{Ker}\,f$ for some n and, hence, $X \in \mathrm{C}_{\wedge^n\operatorname{Ker}\,f}$. Therefore, if we prove

$$\text{If } X \in \mathrm{C}_{\wedge^n\operatorname{Ker}\,f} \text{ then } X \square_C M \in \mathcal{F}_{\operatorname{Ker}\,g} \tag{13}$$

then (12) holds in the finite-dimensional case. We proceed by induction on n. If $n = 1$ and $X \in \mathrm{C}_{\operatorname{Ker}\,f}$ then, by (10), $\rho_{X\square_C M} = 0$. By the version for ρ of (8), $X \square_C M = \operatorname{Ker}\,\rho_{X\square_C M} \in \mathrm{C}_{\operatorname{Ker}\,g}$. By Lemma 4.3, (13) holds for $n = 1$. Let $n > 1$ and $X \in \mathrm{C}_{\wedge^n\operatorname{Ker}\,f}$. By (11) and the induction hypothesis, $X \square_C f(\bigwedge^n \operatorname{Ker}\,f) \square_C M \in \mathcal{F}_{\operatorname{Ker}\,g}$. By (10), $\operatorname{Im}\rho_{X\square_C M} \subseteq X \square_C f(\bigwedge^n \operatorname{Ker}\,f) \square_C M$. Therefore, $\operatorname{Im}\rho_{X\square_C M} \in \mathcal{F}_{\operatorname{Ker}\,g}$. Since $\operatorname{Ker}\,\rho_{X\square_C M}$ belongs to $\mathrm{C}_{\operatorname{Ker}\,g}$, it follows that $\operatorname{Ker}\,\rho_{X\square_C M} \in \mathcal{F}_{\operatorname{Ker}\,g}$ (Lemma 4.3). Since $\mathcal{F}_{\operatorname{Ker}\,g}$ is closed under extensions, we have proved (13) and, therefore, (12) in the case that X is finite-dimensional. If X is any right C-comodule, then we can write X as direct union of finite-dimensional subcomodules. Moreover, $-\square_C M$ is a direct limit preserving left exact functor. This entails that $X \square_C M$ can be expressed as direct union of subcomodules which belong to $\mathscr{F}_{\mathrm{Ker}\ g}$. It is not difficult now to see that $X \square_C M$ is itself r_g-torsion free. This finishes the proof of (12).

The last part of the proof of Theorem 4.2 consist in checking that a right C-comodule X is η-coaccesible if and only if $\mathrm{Ann}_X(\mathrm{Ker}\ f)^{\perp}=0$. For a right C-comodule X, let us denote by $t_f(X)$ the greatest subcomodule of X that belongs to $\mathrm{C}_{\mathrm{Ker}\, f}$. By [5, Theorem 4.2], $\mathrm{Ann}_X((\mathrm{Ker}\ f)^{\perp})=0$ if and only if $t_f(X)=0$. We claim that $t_f(X)=\mathrm{Ker}\ \eta_X$ for every right C-comodule X. Once this claim is proved, the proof is finished. We know that $\mathrm{Ker}\,\eta_X \in \mathrm{C}_{\mathrm{Ker}\ f}$, so that $\mathrm{Ker}\ \eta_X \subseteq t_f(X)$. Since η is a natural transformation, the following square commutes:

$$\begin{array}{ccc} t_f(X) & \longrightarrow & X \\ \downarrow{\scriptstyle \eta_{t_f(X)}} & & \downarrow{\scriptstyle \eta_X} \\ t_f(X) \square_C M \square_D N & \longrightarrow & X \square_C M \square_D N \end{array}$$

where the horizontal maps are monomorphisms. Therefore,

$$\mathrm{Ker}\ \eta_{t_f(X)} = \mathrm{Ker}\ \eta_X \cap t_f(X).$$

Now, $\mathrm{Ker}\ \eta_{t_f(X)} = t_f(X)$ and, thus, $t_f(X) \subseteq \mathrm{Ker}\ \eta_X$. This entails the equality $t_f(X) = \mathrm{Ker}\,\eta_X$, which finishes the proof. □

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